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TILING CONVEX SETS BY TRANSLATES

BY

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ABSTRACT

Let K be a compact, convex subset of E^d which can be tiled by a finite number of disjoint (on interiors) translates of some compact set Y. Then we may write $K = X + Y$, where X is finite. The possible structures for K, X and Y are completely determined under these conditions.

1. Introduction

Suppose K is a convex body (set with interior points) in E^d, d-dimensional Euclidean space. A *tiling* of K is a representation of K as the union of sets $K_1, \cdots, K_n$, which are disjoint on interiors. We wish to investigate the problem of tiling K by translates, that is, by K_i which are all translates of each other. Several results of this type are known, the most general being due to Groemer [1], [2], who solved the problem for the case when each K_i is convex, and Stein [5], who dealt with the case when K is a cube. In this paper we solve the problem completely for compact, convex bodies K.

It turns out to be useful to phrase the problem in terms of vector sums, where the vector sum $U + V$ is $\{u + v : u \in U, v \in V\}$. In particular, note that if $U = \{u_0\}$, $U + V$ is simply a translate of V by u_0.

The following notation will be fixed for the remainder of the paper. K is a compact, convex body in E^d; $K = X + Y$ where X is finite, Y is some set satisfying $Y = \mathrm{cl}(\mathrm{int}\, Y)$ and the translates $\{x_i + Y : x_i \in X\}$ are disjoint on interiors. The restriction on Y is not terribly critical, but it does allow us to simplify some of the arguments. We will also assume that a coordinating system is used so that K, X and Y all lie in the halfspace $\{e = (\varepsilon_1, \cdots, \varepsilon_d) : \varepsilon_1 \geqq 0\}$.

In general we will use the standard terminology about convex sets without definition, referring the reader to Grünbaum's book [3] in case of doubt. One easily-defined special term we will use is that of *d-box*—the affine image of a d-dimensional cube. Two d-boxes are *parallel* if their facets are parallel. We will

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also say that the sets R and S are *oblique* if their affine hulls meet in a single point.

Let $Z(p,q)$ denote the set of positive integers $\{0, p, \cdots, (q-1)p\}$. Then if the integer n is the product $p_1 \times p_2 \times \cdots \times p_s$ (where the p_i are positive, but not necessarily prime or in increasing order), it is easy to show that

$$Z(1,n) = Z(1,p_1) + Z(p_1,p_2) + Z(p_1\ p_2,p_3) + \cdots + Z(p_1 \times \cdots \times p_{s-1},p_s),$$

and that each integer is represented in a unique way. Moreover, C. T. Long [4] has shown the converse—that the above is the *only* way to write the set $Z(1,n)$ in a unique way. More precisely, he showed:

(1.1) *Suppose the set* $Z(1,n) = A_1 + A_2 + \cdots + A_s$ *where the* A_i *are sets of non-negative integers and that each integer* m, $0 \leqq m < n$, *can be written uniquely in the form* $a_1 + \cdots + a_s$ *where* $a_i \in A_i$. *Then there exists a factorization of* $n = p_1 \times \cdots \times p_s$ *such that, after rearrangement if necessary,*

$$A_1 = Z(1,p_1),\ A_2 = Z(p_1,p_2), \cdots, A_s = Z(p_1 \times \cdots \times p_{s-1},p_s).$$

It is these types of decompositions which will be useful to us. If M is the sum of one or more of the $Z(p,q)$, we term M a *Moorish set.* (The name comes from the fact that such a sum has patterns within patterns, much as Moorish decorations.) The sum of the remaining $Z(p,q)$ needed to make up the sum for $Z(1,n)$ will be termed the *complementary Moorish set* to M. If M is a Moorish set, $M + [0,1]$ will be termed a *Moorish block*; it consists of a finite set of intervals.

Suppose $E_1, E_2, \cdots, E_r$ are edges of an r-box, and $M_1, \cdots, M_r$ are Moorish sets contained in $E_1, \cdots, E_r$ respectively. Then $M_1 + \cdots + M_r$ will be termed a *Moorish r-set.* In a similar way, the sum of r Moorish blocks will be called a *Moorish r-block*. Note that a Moorish r-block W is simply the sum of a Moorish r-set, M, and an appropriate r-box, B. A Moorish r-block $W = M + B$ is *complementary* to the Moorish r-set M' if M and M' are sums of complementary Moorish sets. An illustration of K as the sum of two complementary Moorish sets may be seen in Fig. 1, where X is represented by dots and Y by the squares.

One final definition: C is a *t-fold cylinder* over C' if $C = C' + I_1 + \cdots + I_t$ where the I_k are intervals which are oblique to each other and to C'

(1.2) THEOREM. *Let K be a compact, convex body in E^d which is written as $X + Y$, where X is finite,* $Y = \mathrm{cl}(\mathrm{int}\, Y)$, *the sets* $\{x_i + \mathrm{int}\, Y : x_i \in X\}$ *are disjoint, and let j be the dimension of the convex hull of X,* conv(X). *Then X is a Moorish j-set, K is a j-fold cylinder (in directions parallel to the edges of* conv(X)) *over a*

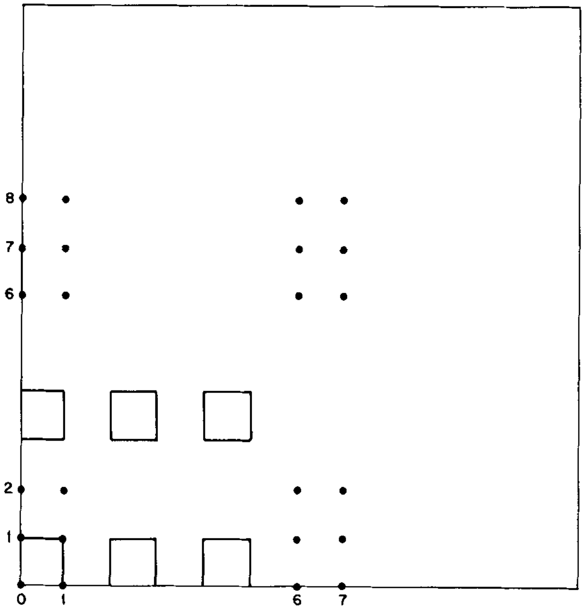

Fig. 1.

convex set K' which is oblique to conv(X), *and Y is the sum of K' and a Moorish j-block complementary to X. Moreover, any two of the three sets K, X, Y uniquely determine the third.*

The above theorem is so complete because the proof is an inductive one and we need most of the hypotheses at each step. The basic strategy is to first solve the case $j = d = 1$ by using Long's result. Then if $j = 1$, $d > 1$, any line through K parallel to conv(X) must break up K as in the case $d = 1$. For $j > 1$ take a facet of conv(X) normal to u and use the observation that the face of K with outer normal u, $F(K, u) = F(X, u) + F(Y, u)$, and use our inductive hypothesis on these faces.

2. The one-dimensional case

X, Y and K are as defined above; in particular, recall that they lie in $\{(\varepsilon_1, \cdots, \varepsilon_d) \colon \varepsilon_1 \geqq 0\}$. Now let $H(r)$ denote the halfspace $\{(\varepsilon_1, \cdots, \varepsilon_d) \colon \varepsilon_1 \leqq r\}$. Then the following assertion is immediate for all dimensions d:

$$K \cap H(r) \subseteq (X \cap H(r)) + (Y \cap H(r)). \tag{2.1}$$

We now restrict ourselves to the case when $K \subseteq E^1$. In this case we will not distinguish between a point and its coordinate. Without loss of generality, we may suppose that $0 \in X \cap Y \cap K$ and that the first two members of X are 0 and 1 (that is, if $x \in X$ and $x < 1$, then $x = 0$). Under these assumptions and since Y is closed, we have the following corollary to (2.1):

(2.2) *The half-open interval* $[0, 1)$ *is contained in* Y. *Hence, the closed interval* $[0, 1] \subseteq Y$.

We may use this to prove the following central result:

(2.3) *Under the assumptions above on* X, Y *and* K, *then* $Y = [0, 1] + Y_0$ *where both* X *and* Y_0 *are sets of integers and* $K = [0, k]$ *where* k *is an integer.*

PROOF. We will show that the initial segments $X(n) = X \cap H(n)$, $Y(n) = Y \cap H(n)$ and $K(n) = K \cap H(n)$ are as described for each integer by induction on n. By (2.2) and our initial assumptions, we know the assertion is true for $n = 1$. We now wish to prove it for $n + 1$.

The assertion could fail only if one of the following happens: 1) $K = [0, k]$ where $n < k < n + 1$; 2) there is a point $x_0 \in X$ such that $n < x_0 < n + 1$; 3) Y contains an interval $[a, b]$ where $n \leqq a < b \leqq n + 1$, but $[n, n + 1] \not\subseteq Y$ (since Y is the closure of its interior, we know Y is the union of segments).

The first case is impossible since $X(n)$, $Y(n)$ will sum to make $K(n + 1)$ an integer. For the second case, let x_0 be the smallest non-integer value of X and choose z so that $n < z < x_0$. Then $z \in x(z) + Y$ where $x(z)$ is an integer. If $x(z) \neq 0$, then $x(z) \geqq 1$. Since $z - x(z) \in Y$, $z - x(z) \in [m, m + 1]$ for some $m < n$. But then $[n, n + 1] = x(z) + [m, m + 1]$ meets $x_0 + [0, 1]$ on interiors. Thus $x(z) = 0$, and hence $[n, z] \subseteq Y$ for all $z < x_0$. But now we derive a contradiction by observing that $1 + [n, z]$ has a non-empty intersection with $x_0 + [0, 1]$. Hence, the second case does not occur.

In the third case, we may as well suppose that if $[a, b] \subseteq [a', b'] \subseteq Y$, then $a = a'$, $b = b'$. Now suppose $n < a$ and choose z so that $n < z < a$. Reasoning as above with $z \in x(z) + Y$ shows $x(z) = 0$ (since $x(z) \leqq n + 1$ already implies $x(z)$ is an integer). Thus the interval $[n, a] \subseteq Y$ and so $[n, b] \subseteq Y$, contrary to our assumption that $[a, b]$ was maximal. Similar reasoning disposes of the subcase when $b < n + 1$ and the assertion is proved.

With the above result at hand, we are now in a position to prove Theorem (1.2) for $d = 1$. It follows from (2.3) that $K = [0, 1] + Y_0 + X$. Put another way, since X and Y_0 are sets of integers, each of the integers $\{0, 1, \cdots, k - 1\}$ is uniquely expressible in the form $x + y$ for some $x \in X$, $y \in Y_0$. But this is exactly the situation covered by (1.1) and so it follows that X is a Moorish set and Y is a Moorish block. In addition, it follows easily that K and X determine

Y_0, and thus Y, and also that K and Y determine X. It is trivial that X and Y determine K.

3. Proof of theorem when $j = 1$

By the previous section our result is known when $d = 1$ and when $j = \dim(\operatorname{aff} X) = 1$. We will proceed by induction on d. So assume the result is known for $j = 1$ whenever $d < d_0$, and suppose $\dim(K) = d_0$.

Let L be some line parallel to $\operatorname{conv}(X)$ which meets K. Consider the set $K_L = K \cap L$. Then $\dim(K_L) < d_0$ and $K_L = X + Y_L$ for some suitable $Y_L \subseteq Y$. By our induction hypothesis, X, Y_L and K_L are as described in (2.3). Moreover, Y_L is uniquely determined by X and K_L and its length is an integer multiple of $\delta(X)$, the distance between the two nearest points of X (0 and 1 in the proof of (2.3)). But the length of Y_L is a continuous function of L (so long as $K_L \neq \emptyset$) and so the length of Y_L is a constant. That is, all line segments $K \cap L$ parallel to $\operatorname{conv}(X)$ have the same length. Since K is convex, it immediately follows that K is a cylinder over some base K^*, and that K^* is convex.

It is trivial that X is a Moorish 1-set by our induction hypothesis. Moreover, each Y_L is a Moorish 1-block uniquely determined by X. Hence Y is a union of translates of any given Y_L and thus $Y = K^* + Y^*$ for some Moorish 1-block Y^*. Finally, since each Y_L is uniquely determined by X, and since Y^* is a translate of the Y_L, Y is uniquely determined by X.

4. Some preliminary results

Before going on with the proof of the general case of the theorem, we need some additional results. This first is a generalization of a result due to Stein [5] which may be proved in exactly the same way.

(4.1) *Let K be a box in E^d with edges which have integral length and suppose K is the union of translates of a fixed set Y which is bounded by a finite number of subsets of hyperplanes parallel to the facets of K at an integer distance from them. Then Y is the product $Y_1 \times \cdots \times Y_d$ where each Y_i lies in an edge of K.*

Our other needed lemma refers to summands of boxes.

(4.2) *Let B, C, D be convex bodies in E^d with $B = C + D$ where B and C are parallel boxes in E^d. Then D is a box parallel to B and C.*

Proof. We first observe that D must be a polytope. For if not, there exists an infinite set U of directions such that $F(C, u) = \{c_0\}$, a vertex of C and

$F(D,u)\neq F(D,u')$ if u, $u'\in U$, $u\neq u'$. But then since $F(C,u)+F(D,u)=F(B,u)$ it would follow that B has an infinite number of vertices, contrary to assumption.

Now let $U(B)$, $U(C)$ and $U(D)$ denote the set of unit outer normals to the facets of B, C and D, respectively. Then it is well known that $U(B)\supseteq U(C)+U(D)$. However, by assumption, $U(B)=U(C)$, hence $U(B)\supseteq U(D)$. But since $U(B)$ consists of d opposite pairs of directions, and since D is bounded, it follows that $U(D)$ must contain each of those pairs. That is, $U(D)=U(B)$. Put differently, the facets of D occur in d parallel pairs. Hence, D is a box.

5. Proof of theorem

For the remainder of the proof we use a double induction on d and j. When $j=d$, we first show that K, $X^*=\operatorname{conv}(X)$ and $Y^*=\operatorname{conv}(Y)$ are all parallel boxes. Then, after establishing that the points of X lie on the integer lattice, we may apply Stein's result to get the bulk of the theorem. After this, the general case follows by taking intersections with K by j-flats parallel to X (a *flat* is a translate of a subspace).

Part 1. Suppose $j=d=d_0>1$ and suppose the result is known for all values of d whenever $j<d_0$. Our initial aim is to show that K, X^* and Y^* are all parallel boxes.

Let $F(X^*,u)$ be any facet of X^* lying in the hyperplane $H(X^*,u)$. Now let H be a hyperplane parallel to $H(X^*,u)$ and so near to it that no members of X lie in the open slab between H and $H(X^*,u)$. If H^+ denotes that closed halfspace determined by H which contains $H(X^*,u)$, then it follows from (2.1) that, after translation if necessary, $H^+\cap K=F(X,u)+(H^+\cap Y)$. Now we apply our induction hypothesis to see that $H^+\cap K$ is a $(d-1)$-fold cylinder over a 1-dimensional convex set; that is, $H^+\cap K$ is a d-box. Since all but one of the facets of $H^+\cap K$ are subsets of facets of K, it thus follows that all facets of K which meet $F(K,u)$ occur in parallel pairs.

Now let S_1, S_2 be two "opposite" subfacets (that is, faces of dimension $d-2$) of $F(X^*,u)$. We may suppose that S_i lies in the facets $F(X^*,u)$ and $F(X^*,v_i)$. Apply the argument above to both facets $F(X^*,v_1)$ and $F(X^*,v_2)$. Then all facets of K which meet $F(K,v_1)$ or $F(K,v_2)$ also occur in parallel pairs. Projecting K orthogonally (call the map π) onto the two-dimensional subspace orthogonal to S_1 and S_2, it is not hard to see that one of these pairs of facets meeting $F(K,v_1)$ must include $F(K,u)$. Similarly for $F(K,v_2)$. In particular,

$\pi F(K, v_1)$, $\pi F(K, u)$, $\pi F(K, v_2)$ and $\pi F(K, -u)$ are the edges of a parallelogram. Thus, $v_1 = -v_2$. This is true for all pairs of opposite subfacets of $F(X^*, u)$, so all facets of X^* meeting $F(X^*, u)$ also occur in parallel pairs, which are respectively parallel to those facets of K meeting $F(K, u)$.

Continuing, we begin the argument over again for $F(X^*, v_1)$ and observe that all facets of X^* meeting $F(X^*, v_1)$ occur in parallel pairs and one of the pairs includes $F(X^*, u)$—and hence $f(X^*, -u)$.

Since $F(X^*, u)$ is a $(d-1)$-box, and since every facet of X^* meeting $F(X^*, u)$ in a subfacet also meets $F(X^*, -u)$ in a subfacet, it follows that X^* is a cylinder over $F(X^*, u)$. That is, X^* is a box. The same reasoning shows that K is a box parallel to X^*. From (4.2), it follows that Y^* is also a box parallel to K and X^*.

Part 2. We are still assuming $j = d = d_0 > 1$ and that the theorem has been proved for all values of d whenever $j < d_0$. By the above we now know that K, X^*, Y^* are parallel boxes. From this fact, it is easy to deduce that all of the edges incident to corresponding vertices of K, X^* and Y^* are parallel.

Thus, by a suitable choice of coordinate systems, and translations if necessary, we may suppose that K, X and Y all lie in the positive orthant

$$\{(\varepsilon_1, \cdots, \varepsilon_d): \varepsilon_i \geqq 0\},$$

that the origin lies in each, that each axis of the coordinate system contains an edge of K, and that the points $(0, \cdots, 1, 0, \cdots, 0)$ are each first members of X along the various axes,—that is, if $(0, \cdots, \alpha, 0, \cdots, 0) \in X$, and $\alpha < 1$, then $\alpha = 0$. Under these hypotheses, we wish to show that every member of X lies on the integer lattice.

We may order the $x \in X$ lexicographically, that is, $(\xi_1, \cdots, \xi_d) \leqq (\eta_1, \cdots, \eta_d)$ iff $\xi_i = \eta_i$, $i = 1, \cdots, k-1$, $\xi_k \leqq \eta_k$. With respect to this ordering, choose the first member of X, if it exists, which is not on the integer lattice. Call it $x_0 = (\xi_{01}, \cdots, \xi_{0d})$. Note that $y_1 = (\xi_{01}, \xi_{02} + \alpha, \cdots, \xi_{0d} + \alpha)$ is a boundary point between $x_0 + Y$ and $x_1 + Y$ for some $x_1 < x_0$ if $0 < \alpha < 1$. Choose α so that a $(d-1)$-disc around y_1 still belongs to $x_1 + Y$. Then $y_1 - x_1$ is a boundary point of Y. We will term it a ξ_1^--boundary point since Y lies in the $-\xi_1$ direction from $y_1 - x_1$. In an analogous way define ξ_1^+-boundary points.

But then $y_1 - x_1$ is a ξ_1^+-boundary point of $x_2 + Y$ where $x_2 \leqq x_0$ if $y_1 - x_1$ is an interior point of K. And thus $y_1 - x_1 - x_2$ is a ξ_1^+-boundary point of Y. This process may be continued only a finite number of times before $y_1 - x_1 - x_2 - \cdots - x_r$ is a ξ_1^--boundary point of K. But by hypothesis, each of the x_i lie on the integer lattice and since $y_1 - x_1 - \cdots - x_r$ has ξ_1-coordinates of 0, it follows that y_1, and hence x_0, has an integer ξ_1-coordinate.

In a similar way by considering the boundary point $y_2 = (\xi_{01} + \alpha, \xi_{02}, \xi_{02} + \alpha, \cdots, \xi_{0d} + \alpha)$, we may show that ξ_{02} is an integer. One must be somewhat careful here to choose α so small that only members of X with integer coordinates lie before y_2. Once this is done, however, the proof above works. Then we may dispose of $\xi_{03}, \cdots, \xi_{0d}$ in that order to conclude that x_0 does lie on the integer lattice, contrary to hypothesis.

Hence, every member of X lies on the integer lattice, as we wished to show. From this, it is almost immediate that Y is a union of integer lattice translates of the corner block, that is, the box $B = \{(\varepsilon_1, \cdots, \varepsilon_d)\colon 0 \leqq \varepsilon_i \leqq 1\}$. It also then follows that each edge of K has integer length.

We now see that all of the hypotheses needed to apply to Stein's result (4.1) are satisfied. Hence, we can conclude that Y is a direct product of the sets which lie along the edges of Y^*. Since each of these sets is known to be a Moorish block, it follows that Y is a Moorish block.

But then $Y = Y_0 + B$, where B was defined above and Y_0 is a subset of the integer lattice. So $K = X + Y_0 + B$, or $(X + B)$ tiles K. Thus, as above, $X + B$ is a Moorish block. Factoring out B, we are left with the result that X is a Moorish set complementary to Y_0.

Since X and Y uniquely determine each other in the 1-dimensional case, and since the d-dimensional case is simply a direct product of the 1-dimensional cases, X and Y uniquely determine each other when $j = d$.

Part 3. Most of the work is now behind us and we are in a position to conclude the proof. In this portion of the argument we assume that $j_0 < d_0$, that the result is known for all values of d if $j < j_0$ and for values of $d < d_0$ when $j = j_0$. We wish to establish the result for $j = j_0$ and $d = d_0$. The idea is precisely as in Section 3 which dealt with the case $j = 1$.

Let F be a j-dimensional flat containing X^* and let F_1 be a parallel flat meeting K. Then, as in the 1-dimensional case, it is clear that $F_1 \cap K = X + (F_2 \cap Y)$ if F_2 is another suitably-chosen parallel j-flat. Moreover, $F_1 \cap K$ is a j-dimensional convex set. Hence, by our induction hypothesis, X^*, $F_1 \cap K$ and $\operatorname{conv}(F_2 \cap Y)$ are boxes. In particular, if $e_1, \cdots, e_j$ represent a set of affinely independent edges of $\operatorname{conv}(X)$, then $F_1 \cap K$ is a cylinder in each of these directions. Moreover, the length of $F_1 \cap K$ in direction e_i is a continuous function of F_1 and is also an integer if the coordinates correspond to X properly (as they were set in Part 2). Hence, the length of $F_1 \cap K$ in direction e_i is constant. Since this is true for all F_1, it follows that K is a cylinder over, say, a set K_1 in each direction $e_1, \cdots, e_j$. It is easy to see K_1 is convex and that K_1 is oblique to each of the e_i.

Moreover, by a similar reasoning, all the $F_2 \cap Y$ are translates of each other and Y is also a j-fold cylinder in directions $e_1, \cdots, e_j$ over K_1. In addition, since each two of $F_1 \cap K$, X and $F_2 \cap Y$ determine each other uniquely, so do K, X and Y.

This completes the proof.

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The result of Grant used in Proposition 1.1 is false [2]. But the proposition can be established as follows. It is known that for $\tilde{A}$, a countable atomless Boolean algebra, $\mathrm{Aut}(\tilde{A})$ is simple and uncountable [1]. A result of Marsh implies that the definable automorphisms are a normal subgroup of $\mathrm{Aut}(\tilde{A})$. The uncountability and simplicity of $\mathrm{Aut}(\tilde{A})$ show that the definable automorphisms of $\tilde{A}$ are trivial. Hence the automorphisms of its generic copy A are trivial. We wish to thank F. D. Hammer for informing us of the papers of Ziegler and Monk.

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